

where  $t_{11}^+$  and  $t_{11}^-$  are values of the stress on the two sides of the shock. In this form, the first two equations express the thermodynamic variables  $\rho$  and  $t_{11}$  in terms of the kinematical quantities  $u$  and  $U$ . These equations can alternatively be written to express these latter quantities in terms of the former:

$$U^2 = \frac{1}{\rho_R} \frac{[t_{11}]}{[\rho_R/\rho]}, \quad [u]^2 = \frac{1}{\rho_R} [t_{11}] [\rho_R/\rho]. \quad (2.15)$$

Equation (2.14)<sub>3</sub>, often called the *Rankine-Hugoniot equation*, has the important property that it involves only the thermodynamic variables.

The jump conditions (2.13)<sub>1,2</sub> can be arranged in the form  $[-t_{11}] = \rho_R U [u]$ ,  $[-t_{11}] = -(\rho_R U^2 [v])$  to show that the jump in stress is proportional to the jump in particle velocity through the coefficient  $\rho_R U$  and to the jump in specific volume through the coefficient  $-(\rho_R U)^2$ . Lines in the  $([-t_{11}], [u])$  and  $([-t_{11}], [v])$  planes that connect states preceding and following a shock transition are called *Rayleigh lines* and have slopes corresponding to these coefficients. The quantity  $\rho_R U$  itself is called the *shock impedance* of the material and is a measure of the difficulty with which it is compressed.

The shock jump equations are normally interpreted as constraints on the allowable discontinuous fields. They have an alternate interpretation, however, in terms of steady waves. Indeed, it is easy to show that the jump relations hold between any two points of a smooth waveform that is propagating steadily at a velocity  $U$  (see, for example, [73S7, 74H2]).

If the form of a smooth steady wave can be recorded experimentally, application of the jump relations to a succession of points on this waveform is sufficient to determine the entire deformation path in the  $(t_{11}, \rho)$  and  $(\varepsilon, \rho)$  planes traversed by a material point during passage of the wave. This procedure has been applied by Johnson and Barker [69J1] and Prieto and Renero [73P2] to a study of the viscoplastic behavior of an aluminum alloy and by Schuler [70S1] to the study of a viscoelastic polymer, poly(methylmethacrylate). In many cases recorded waveforms are not steady, and some similarly rigorous, deductive means for interpreting these observations would be most useful. Such a method, based on equations (2.12), has been proposed by Fowles and Williams [70F1] and further developed by Cowperthwaite and Williams [71C7] and Seaman [74S2]. This method is extremely attractive from the theoretical point of view but has so far proven difficult to implement experimentally.

## 2.2. The Hugoniot curve

*Propagation of plane shocks.* If the reference density,  $\rho_R$ , and the state  $\mathcal{S}^+ = \{\varepsilon^+, t_{11}^+, u^+, \rho^+\}$  of the material into which the shock is advancing are specified, there remain five unknown variables,  $U$  and  $\mathcal{S}^- = \{\varepsilon^-, t_{11}^-, u^-, \rho^-\}$  describing the shock. Specification of the boundary loading producing the shock establishes one of these variables. The jump conditions, say in the form (2.14), comprise three relations among the remaining four variables. One additional relationship is needed to complete the determination of all five variables. This additional relationship is called a *Hugoniot curve* and reflects the differing behaviors of specific materials. Measurement of Hugoniot curves forms one of the major tasks of shock-compression research.

The most commonly measured of the variables are the shock velocity  $U$  and the particle velocity  $u^-$ . The locus of associated values of  $U$  and  $u^-$  determined from a set of experiments involving

shocks of varying strength defines a Hugoniot curve. This curve depends on the state  $\mathcal{S}^+$  of the material into which the shock is propagating and is said to be *centered* on this state. The Hugoniot curve centered on the state of rest at normal laboratory ambient conditions is called the *principal* Hugoniot curve of the material in question.

Experimental measurements are normally made relative to coordinates chosen so that  $u^+ = 0$  and  $U > 0$ . When this choice is made we write  $u^- \equiv u$ . It has been found experimentally that some hundreds of materials are described to good approximation over the entire range of the available data by the linear relation between  $U$  and  $u$ :

$$U = a + bu \quad (2.16)$$

where  $a$  and  $b$  are positive constants characteristic of the material and of the initial state  $\mathcal{S}^+$ , in which the material is at rest under conditions of atmospheric pressure and temperature. By combining a measured Hugoniot curve with the jump conditions, a number of other Hugoniot relationships (ten in all) can be obtained between pairs of variables in the set  $\{U, \mathcal{S}\}$ . In the case where eq. (2.16) holds, these relationships include

$$[-t_{11}] = \rho_R(a + bu)u, \quad [t_{11}] = (\rho_R a)^2 [v] (1 + b\rho_R [v])^{-2}, \quad (2.17)$$

and seven other less frequently used relationships. Because shocks can propagate in either the  $+X$  or  $-X$  direction, Hugoniot curves involving shock or particle velocity have two branches. Specifically, a Hugoniot curve relating stress and particle velocity jump can be reflected and translated in  $u$  so that a given jump in stress can be obtained for a wave having either positive or negative velocity and/or propagating into material in uniform motion. Information about the Hugoniot curve is the normal outcome of a series of shock-compression experiments. Hugoniot curves for three representative metals are shown in fig. 2.1.

It is significant that the foregoing discussion has been concerned entirely with mechanical variables and has not involved temperature or entropy. These quantities increase upon shock compression of a body but are not normally measured experimentally and cannot be calculated without invoking additional assumptions. Restricting attention to fluids, we see that when the Rankine-Hugoniot equation (2.14)<sub>3</sub> is combined with the first law of thermodynamics the differential equations

$$\frac{ds_H}{dv} = \frac{1}{2\theta_H} \left[ (v^+ - v) \frac{dp_H}{dv} + p_H - p^+ \right], \quad \frac{d\theta_H}{dv} = -\frac{\gamma}{v} \theta_H + \frac{1}{2C_v} \left[ (v^+ - v) \frac{dp_H}{dv} + p_H - p^+ \right] \quad (2.18)$$

for calculating the entropy and temperature of Hugoniot states can be obtained. In these relations  $p_H(v)$ ,  $s_H(v)$  and  $\theta_H(v)$  are the pressure, entropy density, and temperature along the Hugoniot curve centered on  $\mathcal{S}^+$ ,  $\gamma \equiv v(\partial p/\partial \varepsilon)_v$  is Grüneisen's parameter, and  $C_v$  is the specific heat at constant volume. Evaluation of  $\theta_H$  and  $s_H$  requires that  $\gamma$  and  $C_v$  be known at all states on the Hugoniot curve. Expansion of  $s_H(v)$  about the point  $v = v^+$  gives, in consideration of eq. (2.18)<sub>1</sub>, the result

$$[s] = -\frac{1}{12\theta_H} \left( \frac{d^2 p_H}{dv^2} \right)_{v^+} [v]^3 + \dots \quad (2.19)$$

From this relation we see that the increase in entropy of an element of material upon passage of a shock is of third order in the compression, i.e., small for small compressions. The Hugoniot curve